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LETTER TO THE EDITOR

Equivalence of an Ising model with two- and three-spin interactions with the four-state Potts model

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Abstract. The Ising model on the square lattice with two-spin interactions in the y direction and three-spin interactions in the x direction is related to the four-state Potts model on the square lattice. This mapping becomes exact in the limit of strong three-spin couplings.

We consider an Ising model on the square lattice with two-spin interactions in the vertical direction and with three-spin interactions in the horizontal direction. The reduced Hamiltonian of an $m \times 3n$ system is

$$H^{(n)}/kT = -\sum_{i,j} (K_2 s_{i-1,j} s_{i,j} + K_3 s_{i,j-2} s_{i,j-1} s_{i,j}) \tag{1}$$

with $K_2 \geqslant 0$, $K_3 \geqslant 0$, $S_{i,j} = \pm 1$, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, 3n$ denote the vertical and horizontal coordinates respectively. This model is self-dual (see, e.g., Gruber *et al* 1977). If a single phase transition exists for constant ratio K_2/K_3, then it must be located at the self-dual point determined by

$$\sinh(2K_2)\sinh(2K_3) = 1. \tag{2}$$

The model has already received considerable attention in recent years (Penson *et al* 1982, Debierre and Turban 1983, Iglói *et al* 1983, 1986, Vanderzande and Iglói 1987). Parts of these references treat the anisotropic limit $K_2 \to \infty$, $K_3 \to 0$. The results confirm that there is indeed a phase transition at the self-dual point. It has been suggested that the phase transition is of the four-state Potts type. This suggestion was supported by Horiguchi and Goncalves (1985) who constructed a mapping between the Baxter–Wu model and a model resembling that described by equation (1). But numerical analyses of the model (1), using finite-size scaling and other methods, were found to produce estimates of the temperature exponent $y_T \approx \frac{4}{3}$, which is too low in comparison with the exact value $y_T = \frac{3}{2}$ for the four-state Potts model (den Nijs 1981, Black and Emery 1981). However, a finite-size analysis of small four-state Potts models (Blöte and Nightingale 1982) also yielded estimates considerably below $y_T = \frac{3}{2}$. These unusually large deviations could be satisfactorily explained by the presence of a marginal operator (Nienhuis *et al* 1979), leading to logarithmic terms in the finite-size expansion of the free energy (Blöte and Nightingale 1982, Cardy 1986).

Thus the numerical results for the temperature exponent of the model (1) are not inconsistent with four-state Potts universality, but so far they have failed to provide a trustworthy confirmation of this classification. Under these circumstances it remains of interest to construct analytic evidence to answer this problem. For this purpose, we take the opposite limit to the one originally considered by Penson *et al* (1982):

$$\exp(-2K_3) = \varepsilon \qquad K_2 = \mu\varepsilon \qquad \varepsilon \to 0$$

where μ^{-1} is a temperature-like variable. Duality (2) predicts the critical value $\mu_c = 1$. The mapping on the four-state Potts model is established by combining all triplets $(s_{i,3j-2}, s_{i,3j-1}, s_{i,3j})$ and executing the sum in the partition function over all $s_{i,3j-2}$. The remaining spins are condensed into variables $\sigma_{i,j}$ which can assume four possible values.

A convenient way to show that the resulting interactions between the variables $\sigma_{i,j}$ are of the four-state Potts type uses the transfer matrix in analogy with the second description by Schultz *et al* (1964). Consider a rectangular model (1) with dimensions m and $3n$, free boundaries in the horizontal direction and periodic boundaries in the vertical direction: $s_{0,j} \equiv s_{m,j}$ (although this is not essential). Columns of spins $s_{i,j}(i = 1, 2, \ldots, m)$ are denoted $\boldsymbol{s}_j$. The partition sum $Z^{(n)}$ is divided into 2^{2m} restricted sums:

$$Z^{(n)}(\boldsymbol{s}_{3n-1}, \boldsymbol{s}_{3n}) = \sum_{\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_{3n-2}} \exp[-H^{(n)}/kT]$$

with $H^{(n)}$ defined by (1). The following relation exists between these restricted sums for systems with lengths $3n$ and $3(n+1)$:

$$Z^{(n+1)}(\boldsymbol{\sigma}_{n+1}) = \sum_{\boldsymbol{\sigma}_n} T(\boldsymbol{\sigma}_{n+1}, \boldsymbol{\sigma}_n) Z^{(n)}(\boldsymbol{\sigma}_n) \tag{3}$$

where $\mathbf{T}$ is the transfer matrix, and the spin variables are written in terms of

$$\boldsymbol{\sigma}_j = (\sigma_{1j}, \sigma_{2j}, \ldots, \sigma_{mj})$$

where

$$\sigma_{ij} = s_{i,3j-1} + \tfrac{1}{2} s_{i,3j} + \tfrac{5}{2} \tag{4}$$

can assume the values 1, 2, 3 and 4. The transfer matrix is given by

$$T(\boldsymbol{\sigma}_{n+1}, \boldsymbol{\sigma}_n) = \sum_{\boldsymbol{s}_{3n+1}} \exp[-(H^{(n+1)} - H^{(n)})/kT]. \tag{5}$$

The interactions contributing to $\mathbf{T}$ are the $3m$ two-spin interactions in columns $3n+1$ to $3n+3$, and the $3m$ three-spin interactions between the spins in columns $3n-1$ to $3n+3$ (see figure 1). As a consequence of the anisotropic limit, we need consider, apart from the leading contributions to $\mathbf{T}$, only corrections which are first order in ε (Kogut 1979). The leading contributions are equal to $\exp[3mK_3] = \varepsilon^{-3m/2}$, and are located on the diagonal of $\mathbf{T}$: no spin flip. Only one term in the summation over $\boldsymbol{s}_{3n+1}$ survives in each diagonal element of $\mathbf{T}$, namely the term with

$$s_{i,3n-1} s_{i,3n} s_{i,3n+1} = 1 \tag{6}$$

for all i. All other terms are of too high an order in ε. The surviving terms also contain contributions of relative order ε, due to the weak couplings K_2. These contributions are

$$\mu\varepsilon^{1-3m/2} \sum_{i=1}^{m} (s_{i-1,3n+1} s_{i,3n+1} + s_{i-1,3n+2} s_{i,3n+2} + s_{i-1,3n+3} s_{i,3n+3}).$$

Using (4) and (6), the diagonal contributions of relative order ε can also be written

$$\mu\varepsilon^{1-3m/2} \sum_{i=1}^{m} (4\delta_{\sigma_{i,n}\sigma_{i+1,n}} - 1).$$

The leading off-diagonal terms in $\mathbf{T}$ are due to a single spin flip, i.e. $\sigma_{i,n} \neq \sigma_{i,n+1}$ for only one value of i. Given such a spin flip, only one term of sufficiently low order in ε survives the summation over $\boldsymbol{s}_{3n+1}$. Its value is $\varepsilon^{1-3m/2}$, independent of the remaining degrees of freedom of the σ variables. Combining all these contributions yields

$$T(\boldsymbol{\sigma}_{n+1}, \boldsymbol{\sigma}_n) = \varepsilon^{-3m/2}(1 - m\mu\varepsilon)[\mathbb{1} + \varepsilon t(\boldsymbol{\sigma}_{n+1}, \boldsymbol{\sigma}_n)] \tag{7}$$

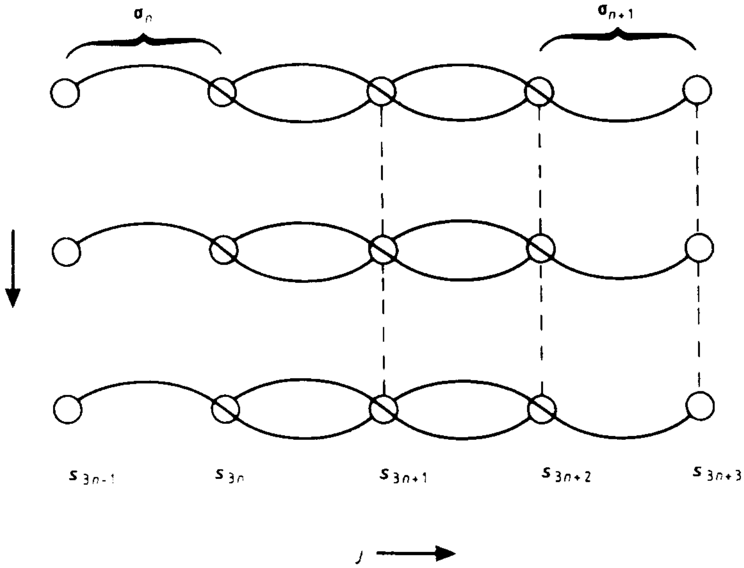

Figure 1. Interactions contributing to the transfer matrix **T** defined in the text. Vertical two-spin couplings K_2 are indicated by broken lines. Horizontal couplings K_3 are shown by curved lines, each of which connects three spins, intersecting the middle one. Columns of spins $s_{i,j}$ are denoted s_j. Two neighbouring columns s_{3k-1} and s_{3k} are combined into one column of Potts variables $\boldsymbol{\sigma}_k$.

where

$$t(\boldsymbol{\sigma}_{n+1}, \boldsymbol{\sigma}_n) = 4\mu\delta_{\boldsymbol{\sigma}_n\boldsymbol{\sigma}_{n+1}} \sum_{i=1}^{m} \delta_{\sigma_{i,n}\sigma_{i+1,n}} + \sum_{i=1}^{m} (1 - \delta_{\sigma_{i,n}\sigma_{i,n+1}}) \prod_{k\neq i} \delta_{\sigma_{k,n}\sigma_{k,n+1}}. \tag{8}$$

Next, consider the Potts model described by

$$H^{(n)}_{\text{Potts}} = \sum_{i,j} K_x \delta_{\tau_{i,j}\tau_{i,j+1}} + K_y \delta_{\tau_{i,j}\tau_{i+1,j}} \tag{9}$$

with $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, and $\tau_{ij} = 1, 2, 3$ or 4, in the anisotropic limit

$$\exp(-K_x) = \varepsilon \qquad K_y = 4\mu\varepsilon \qquad \varepsilon \to 0$$

where μ^{-1} plays the role of the temperature variable. Columns of Potts variables $\tau_{i,j}$, $i = 1, 2, \ldots, m$ are taken together as $\boldsymbol{\tau}_j$. The transfer matrix which adds a column $\boldsymbol{\tau}_{n+1}$ to the model (9) is

$$T_{\text{Potts}}(\boldsymbol{\tau}_{n+1}, \boldsymbol{\tau}_n) = \varepsilon^{-m}[\mathbb{1} + \varepsilon t(\boldsymbol{\tau}_{n+1}, \boldsymbol{\tau}_n)] \tag{10}$$

where the definition (8) of $\mathbf{t}$ applies also in this case. A comparison with (7) shows that the two transfer matrices differ only by a simple multiplicative factor. Hence, the free energies of the two models (1) and (9) per physical unit of area (one lattice unit in the vertical direction, ε^{-1} units of the Potts model in the horizontal direction, $3\varepsilon^{-1}$ lattice units of the Ising model in the horizontal direction) differ only by a trivial (but divergent) contribution:

$$f_{\text{Ising}} = \frac{3}{2\varepsilon} \ln \frac{1}{\varepsilon} - \mu + \frac{1}{m} \lambda(m, \mu)$$

$$f_{\text{Potts}} = \frac{1}{\varepsilon} \ln \frac{1}{\varepsilon} + \frac{1}{m} \lambda(m, \mu)$$

where $\lambda(m, \mu)$ is the leading eigenvalue of $\mathbf{t}$. The critical behaviour, which resides in $\lambda(m, \mu)$, is identical for both models.

It seems very unlikely that the model (1) would leave the four-state Potts universality class when K_3 becomes finite (and the mapping is no longer exact). In the first place, a change of ratio K_2/K_3 may be associated with a trivial rescaling in one direction, leaving critical exponents invariant. Furthermore, recent extensive Monte Carlo simulations of the model† (1) with $K_2 = K_3$ have accurately confirmed (Blöte *et al* 1986) the four-state Potts nature of the phase transition.

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† This model is called the ATTTI (anisotropic two times three Ising) model by the group operating the DISP, the special purpose computer used for these simulations.